CrossMark

A multiplicative property characterizes quasinormal composition operators in L^2-spaces

Piotr Budzyński [a], Zenon Jan Jabłoński [b], Il Bong Jung [c], Jan Stochel [b,*]

[a] *Katedra Zastosowań Matematyki, Uniwersytet Rolniczy w Krakowie, ul. Balicka 253c, PL-30198 Kraków, Poland*
[b] *Instytut Matematyki, Uniwersytet Jagielloński, ul. Łojasiewicza 6, PL-30348, Kraków, Poland*
[c] *Department of Mathematics, Kyungpook National University, Daegu 702-701, Republic of Korea*

ARTICLE INFO

ABSTRACT

A densely defined composition operator in an L^2-space induced by a measurable transformation ϕ is shown to be quasinormal if and only if the Radon–Nikodym derivatives h_{ϕ^n} attached to powers ϕ^n of ϕ have the multiplicative property: $\mathsf{h}_{\phi^n} = \mathsf{h}_\phi^n$ almost everywhere for $n = 0, 1, 2, \dots$.

1. Introduction

Composition operators (in L^2-spaces over σ-finite measure spaces) play an essential role in ergodic theory. They are also interesting objects of operator theory. The foundations of the theory of bounded composition operators are well-developed. In particular, the questions of their boundedness, normality, quasinormality, subnormality, seminormality etc. were answered (see e.g., [21,19,26,12,15,16,9,11,22,6] for the general approach and [10,17,23,8,24] for special classes of operators; see also the monograph [22]).

As opposed to the bounded case, the theory of unbounded composition operators is at a rather early stage of development. There are few papers concerning this issue. Some basic facts about unbounded composition operators can be found in [7,13,4]. In a recent paper [5], we gave the first ever criterion for subnormality of unbounded densely defined composition operators, which states that if such an operator admits a measurable family of probability measures that satisfy the consistency condition (see (CC)), then it is subnormal (cf. [5, Theorem 9]). The aforesaid criterion becomes a full characterization of subnormality in the bounded case. Recall that the celebrated Lambert's characterization of subnormality of bounded composition operators (cf. [15]) is no longer true for unbounded ones (see [14, Theorem 4.3.3] and [4, Section 11]). It turns out that the consistency condition is strongly related to quasinormality.

Quasinormal operators, which were introduced by A. Brown in [3], form a class of operators which is properly larger than that of normal operators, and properly smaller than that of subnormal operators (see [3, Theorem 1] and [25, Theorem 2]). It was A. Lambert who noticed that if C_ϕ is a bounded quasinormal composition operator with a surjective symbol ϕ, then the Radon–Nikodym derivatives h_{ϕ^n}, $n = 0, 1, 2, \dots$, (see (2.1)) have the following multiplicative property (cf. [15, p. 752]):

$$\mathsf{h}_{\phi^n} = \mathsf{h}_\phi^n \quad \text{almost everywhere for } n = 0, 1, 2, \dots.$$

The aim of this article is to show that the above completely characterizes quasinormal composition operators regardless of whether they are bounded or not, and regardless of whether ϕ is surjective or not (cf. Theorem 3.1). The proof of

* Corresponding author.
   E-mail addresses: piotr.budzynski@ur.krakow.pl (P. Budzyński), Zenon.Jablonski@im.uj.edu.pl (Z.J. Jabłoński), ibjung@knu.ac.kr (I.B. Jung), Jan.Stochel@im.uj.edu.pl (J. Stochel).

this characterization depends on the fact that a quasinormal composition operator always admits a special measurable family of probability measures which satisfy the consistency condition (CC). This leads to yet another characterization of quasinormality (see condition (iii) of Theorem 3.1).

2. Preliminaries

We write $\mathbb{C}$ for the field of all complex numbers and denote by $\mathbb{R}_+$, $\mathbb{Z}_+$ and $\mathbb{N}$ the sets of nonnegative real numbers, nonnegative integers and positive integers, respectively. Set $\overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{\infty\}$. Given a sequence $\{\Delta_n\}_{n=1}^{\infty}$ of sets and a set Δ such that $\Delta_n \subseteq \Delta_{n+1}$ for every $n \in \mathbb{N}$, and $\Delta = \bigcup_{n=1}^{\infty} \Delta_n$, we write $\Delta_n \nearrow \Delta$ (as $n \to \infty$). The characteristic function of a set Δ is denoted by χ_Δ (it is clear from the context on which set the function χ_Δ is defined).

The following lemma is a direct consequence of [18, Proposition I-6-1] and [1, Theorem 1.3.10]. It will be used in the proof of Theorem 3.1.

Lemma 2.1. *Let $\mathscr{P}$ be a semi-algebra of subsets of a set X and ρ_1, ρ_2 be finite measures[1] defined on the σ-algebra generated by $\mathscr{P}$ such that $\rho_1(\Delta) = \rho_2(\Delta)$ for all $\Delta \in \mathscr{P}$. Then $\rho_1 = \rho_2$.*

Let A be a linear operator in a complex Hilbert space $\mathcal{H}$. Denote by $\mathcal{D}(A)$ and A^* the domain and the adjoint of A (in case it exists). If A is closed and densely defined, then A has a (unique) *polar decomposition* $A = U|A|$, where U is a partial isometry on $\mathcal{H}$ such that the kernels of U and A coincide and $|A|$ is the square root of A^*A (cf. [2, Section 8.1]). A densely defined linear operator A in $\mathcal{H}$ is said to be *quasinormal* if A is closed and $U|A| \subseteq |A|U$, where $A = U|A|$ is the polar decomposition of A. We refer the reader to [3] and [25] for basic information on bounded and unbounded quasinormal operators, respectively.

Throughout the paper $(X, \mathscr{A}, \mu)$ will denote a σ-finite measure space. We shall abbreviate the expressions "almost everywhere with respect to μ" and "for μ-almost every x" to "a.e. $[\mu]$" and "for μ-a.e. x", respectively. As usual, $L^2(\mu) = L^2(X, \mathscr{A}, \mu)$ denotes the Hilbert space of all square integrable complex functions on X with the standard inner product. Let $\phi: X \to X$ be an $\mathscr{A}$-*measurable* transformation of X, i.e., $\phi^{-1}(\Delta) \in \mathscr{A}$ for all $\Delta \in \mathscr{A}$. Denote by $\mu \circ \phi^{-1}$ the measure on $\mathscr{A}$ given by $\mu \circ \phi^{-1}(\Delta) = \mu(\phi^{-1}(\Delta))$ for $\Delta \in \mathscr{A}$. We say that ϕ is *nonsingular* if $\mu \circ \phi^{-1}$ is absolutely continuous with respect to μ. If ϕ is a nonsingular transformation of X, then the map $C_\phi: L^2(\mu) \supseteq \mathcal{D}(C_\phi) \to L^2(\mu)$ given by

$$\mathcal{D}(C_\phi) = \{f \in L^2(\mu): f \circ \phi \in L^2(\mu)\} \quad \text{and} \quad C_\phi f = f \circ \phi \quad \text{for } f \in \mathcal{D}(C_\phi),$$

is well-defined (and vice versa). Call such C_ϕ a *composition operator*. Note that every composition operator is closed (see e.g., [4, Proposition 3.2]). If ϕ is nonsingular, then by the Radon–Nikodym theorem there exists a unique (up to sets of measure zero) $\mathscr{A}$-measurable function $\mathsf{h}_\phi: X \to \overline{\mathbb{R}}_+$ such that

$$\mu \circ \phi^{-1}(\Delta) = \int_\Delta \mathsf{h}_\phi \mathrm{d}\mu, \quad \Delta \in \mathscr{A}. \tag{2.1}$$

It is well-known that C_ϕ is densely defined if and only if $\mathsf{h}_\phi < \infty$ a.e. $[\mu]$ (cf. [7, Lemma 6.1]), and $\mathcal{D}(C_\phi) = L^2(\mu)$ if and only if $\mathsf{h}_\phi \in L^\infty(\mu)$ (cf. [19, Theorem 1]). Given $n \in \mathbb{N}$, we denote by ϕ^n the n-fold composition of ϕ with itself; ϕ^0 is the identity transformation of X. Note that if ϕ is nonsingular and $n \in \mathbb{Z}_+$, then ϕ^n is nonsingular and thus h_{ϕ^n} makes sense. Clearly $\mathsf{h}_{\phi^0} = 1$ a.e. $[\mu]$.

Suppose that $\phi: X \to X$ is a nonsingular transformation such that $\mathsf{h}_\phi < \infty$ a.e. $[\mu]$. Then the measure $\mu|_{\phi^{-1}(\mathscr{A})}$ is σ-finite (cf. [4, Proposition 3.2]). Hence, by the Radon–Nikodym theorem, for every $\mathscr{A}$-measurable function $f: X \to \overline{\mathbb{R}}_+$ there exists a unique (up to sets of measure zero) $\phi^{-1}(\mathscr{A})$-measurable function $\mathsf{E}(f): X \to \overline{\mathbb{R}}_+$ such that

$$\int_{\phi^{-1}(\Delta)} f \mathrm{d}\mu = \int_{\phi^{-1}(\Delta)} \mathsf{E}(f) \mathrm{d}\mu, \quad \Delta \in \mathscr{A}. \tag{2.2}$$

We call $\mathsf{E}(f)$ the *conditional expectation* of f with respect to $\phi^{-1}(\mathscr{A})$ (see [4] for recent applications of the conditional expectation in the theory of unbounded composition operators; see also [20] for the foundations of the theory of probabilistic conditional expectation). It is well-known that

$$\text{if } 0 \leqslant f_n \nearrow f \text{ and } f, f_n \text{ are } \mathscr{A}\text{-measurable, then } \mathsf{E}(f_n) \nearrow \mathsf{E}(f), \tag{2.3}$$

where $g_n \nearrow g$ means that for μ-a.e. $x \in X$, the sequence $\{g_n(x)\}_{n=1}^{\infty}$ is monotonically increasing and convergent to $g(x)$.

Now we state three results, each of which will be used in the proof of Theorem 3.1. The first one provides a necessary and sufficient condition for the Radon–Nikodym derivatives h_{ϕ^n}, $n \in \mathbb{N}$, to have the following semigroup property.

Lemma 2.2 ([4, Lemma 9.1]). *If ϕ is a nonsingular transformation of X such that $\mathsf{h}_\phi < \infty$ a.e. $[\mu]$ and $n \in \mathbb{N}$, then the following two conditions are equivalent:*

(i) $\mathsf{h}_{\phi^{n+1}} = \mathsf{h}_{\phi^n} \cdot \mathsf{h}_\phi$ *a.e. $[\mu]$,*
(ii) $\mathsf{E}(\mathsf{h}_{\phi^n}) = \mathsf{h}_{\phi^n} \circ \phi$ *a.e. $[\mu|_{\phi^{-1}(\mathscr{A})}]$.*

The second result is a basic description of quasinormal composition operators.

---

[1] All measures considered in this paper are assumed to be positive.

Proposition 2.3 ([4, Proposition 8.1]). *If* $\phi: X \to X$ *is nonsingular and* C_ϕ *is densely defined, then* C_ϕ *is quasinormal if and only if* $h_\phi = h_\phi \circ \phi$ *a.e.* $[\mu]$.

Before formulating the third result, we introduce the necessary terminology. We say that a map $P: X \times \mathfrak{B}(\mathbb{R}_+) \to [0, 1]$, where $\mathfrak{B}(\mathbb{R}_+)$ is the σ-algebra of all Borel subsets of $\mathbb{R}_+$, is an $\mathscr{A}$-*measurable family of probability measures* if the set-function $P(x, \cdot)$ is a Borel probability measure on $\mathbb{R}_+$ for every $x \in X$, and the function $P(\cdot, \sigma)$ is $\mathscr{A}$-measurable for every $\sigma \in \mathfrak{B}(\mathbb{R}_+)$. Let ϕ be a nonsingular transformation of X such that $h_\phi < \infty$ a.e. $[\mu]$. An $\mathscr{A}$-measurable family $P: X \times \mathfrak{B}(\mathbb{R}_+) \to [0, 1]$ of probability measures is said to satisfy the *consistency condition* (cf. [5]) if

$$\mathsf{E}(P(\cdot, \sigma))(x) = \frac{\int_\sigma t P(\phi(x), \mathrm{d}\, t)}{h_\phi(\phi(x))} \quad \text{for } \mu\text{-a.e. } x \in X \text{ and every } \sigma \in \mathfrak{B}(\mathbb{R}_+). \tag{CC}$$

As shown in [5, Proposition 10], each quasinormal composition operator C_ϕ has an $\mathscr{A}$-measurable family $P: X \times \mathfrak{B}(\mathbb{R}_+) \to [0, 1]$ of probability measures which satisfies the consistency condition (CC). In fact, such P can always be chosen to be $\phi^{-1}(\mathscr{A})$-measurable. As already mentioned, the consistency condition (CC) leads to a criterion for subnormality of unbounded composition operators (cf. [5, Theorem 9]).

The third result relates moments of an $\mathscr{A}$-measurable family P of probability measures satisfying (CC) to Radon–Nikodym derivatives h_{ϕ^n}, $n \in \mathbb{Z}_+$.

Theorem 2.4 ([5, Theorem 17]). *Let* ϕ *be a nonsingular transformation of* X *such that* $0 < h_\phi < \infty$ *a.e.* $[\mu]$, *and* $P: X \times \mathfrak{B}(\mathbb{R}_+) \to [0, 1]$ *be an* $\mathscr{A}$-*measurable family of probability measures which satisfies* (CC). *Then*

$$h_{\phi^n}(x) = \int_{\mathbb{R}_+} t^n P(x, \mathrm{d}\, t) \quad \text{for } \mu\text{-a.e. } x \in X \text{ and for every } n \in \mathbb{Z}_+.$$

3. The characterization

In this section we provide the main characterization of quasinormal composition operators (see (v) below). We begin by noting that if C_ϕ is quasinormal, then the $\mathscr{A}$-measurable family $P: X \times \mathfrak{B}(\mathbb{R}_+) \to [0, 1]$ of probability measures given by $P(x, \sigma) = \chi_\sigma(h_\phi(x))$ for $x \in X$ and $\sigma \in \mathfrak{B}(\mathbb{R}_+)$ satisfies the consistency condition (CC) (of course, under the assumption that h_ϕ is finite). The consistency condition written for this particular P appears in (iii) below. It is an essential component of the proof of the characterization.

From now on, we adhere to the convention $\infty^0 = 1$.

Theorem 3.1. *Let* ϕ *be a nonsingular transformation of* X *such that* C_ϕ *is densely defined. Then the following six conditions are equivalent*[2]:

  (i) C_ϕ *is quasinormal,*
 (ii) $\chi_\sigma \circ h_\phi \circ \phi \cdot \chi_\sigma \circ h_\phi = \chi_\sigma \circ h_\phi \circ \phi$ *a.e.* $[\mu]$ *for every* $\sigma \in \mathfrak{B}(\mathbb{R}_+)$,
(iii) $\mathsf{E}(\chi_\sigma \circ h_\phi) = \chi_\sigma \circ h_\phi \circ \phi$ *a.e.* $[\mu]$ *for every* $\sigma \in \mathfrak{B}(\mathbb{R}_+)$,
 (iv) $\mathsf{E}(f \circ h_\phi) = f \circ h_\phi \circ \phi$ *a.e.* $[\mu]$ *for every Borel function* $f: \mathbb{R}_+ \to \overline{\mathbb{R}}_+$,
  (v) $h_{\phi^n} = h_\phi^n$ *a.e.* $[\mu]$ *for every* $n \in \mathbb{Z}_+$,
 (vi) $\mathsf{E}(h_\phi) = h_\phi \circ \phi$ *a.e.* $[\mu]$ *and* $\mathsf{E}(h_{\phi^n}) = \mathsf{E}(h_\phi)^n$ *a.e.* $[\mu]$ *for every* $n \in \mathbb{Z}_+$.

Proof. It follows from [7, Lemma 6.1] that $h_\phi < \infty$ a.e. $[\mu]$.

(i) $\Rightarrow$ (iii) Since, by Proposition 2.3, $h_\phi = h_\phi \circ \phi$ a.e. $[\mu]$, we deduce that $\chi_\sigma \circ h_\phi = \chi_\sigma \circ h_\phi \circ \phi$ a.e. $[\mu]$ for every $\sigma \in \mathfrak{B}(\mathbb{R}_+)$, which implies (iii).

(iii) $\Rightarrow$ (iv) Since each Borel function $f: \mathbb{R}_+ \to \overline{\mathbb{R}}_+$ is a pointwise limit of an increasing sequence of nonnegative Borel simple functions, one can show that (iii) implies (iv) by applying the Lebesgue monotone convergence theorem as well as the additivity and the monotone continuity of the conditional expectation (see (2.3)).

(iv) $\Rightarrow$ (iii) Obvious.

(iii) $\Rightarrow$ (ii) In view of (iii) and (2.2), we have

$$\int_{\phi^{-1}(\Delta)} \chi_\sigma \circ h_\phi \mathrm{d}\, \mu = \int_{\phi^{-1}(\Delta)} \chi_\sigma \circ h_\phi \circ \phi \mathrm{d}\, \mu, \quad \Delta \in \mathscr{A}. \tag{3.1}$$

By our assumptions on the measure μ, there exists a sequence $\{X_n\}_{n=1}^\infty \subseteq \mathscr{A}$ such that $\{\mu(X_n)\}_{n=1}^\infty \subseteq \mathbb{R}_+$, $X_n \nearrow X$ as $n \to \infty$ and

$$h_\phi \leqslant k \quad \text{a.e. } [\mu] \text{ on } X_k \text{ for every } k \in \mathbb{N}. \tag{3.2}$$

---

[2] Since $h_\phi < \infty$ a.e. $[\mu]$, the expressions $f \circ h_\phi$ and $f \circ h_\phi \circ \phi$ appearing in (iv) are defined a.e. $[\mu]$. To overcome this disadvantage, one can simply set $f(\infty) = 0$.

Substituting $\Delta = \mathsf{h}_\phi^{-1}(\sigma) \cap X_n$ into (3.1), we see that the following equality holds for all $n \in \mathbb{N}$ and $\sigma \in \mathfrak{B}(\mathbb{R}_+)$,

$$\mu(\phi^{-1}(X_n) \cap (\mathsf{h}_\phi \circ \phi)^{-1}(\sigma) \cap \mathsf{h}_\phi^{-1}(\sigma)) = \mu(\phi^{-1}(X_n) \cap (\mathsf{h}_\phi \circ \phi)^{-1}(\sigma)) < \infty.$$

Hence for all $n \in \mathbb{N}$ and $\sigma \in \mathfrak{B}(\mathbb{R}_+)$,

$$\chi_{\phi^{-1}(X_n)} \cdot \chi_{(\mathsf{h}_\phi \circ \phi)^{-1}(\sigma)} \cdot \chi_{\mathsf{h}_\phi^{-1}(\sigma)} = \chi_{\phi^{-1}(X_n)} \cdot \chi_{(\mathsf{h}_\phi \circ \phi)^{-1}(\sigma)} \quad \text{a.e. } [\mu].$$

Since $\phi^{-1}(X_n) \nearrow X$ as $n \to \infty$, we get (ii).

(ii) $\Rightarrow$ (i) Substituting $\mathbb{R}_+ \setminus \sigma$ into (ii) in place of σ, we get

$$\chi_\sigma \circ \mathsf{h}_\phi = \chi_\sigma \circ \mathsf{h}_\phi \circ \phi \cdot \chi_\sigma \circ \mathsf{h}_\phi \stackrel{\text{(ii)}}{=} \chi_\sigma \circ \mathsf{h}_\phi \circ \phi \quad \text{a.e. } [\mu] \text{ for every } \sigma \in \mathfrak{B}(\mathbb{R}_+).$$

Applying the standard measure-theoretic argument, we deduce that $f \circ \mathsf{h}_\phi = f \circ \mathsf{h}_\phi \circ \phi$ a.e. $[\mu]$ for every Borel function $f : \mathbb{R}_+ \to \overline{\mathbb{R}}_+$. Substituting $f(t) = t$, $t \in \mathbb{R}_+$, we see that $\mathsf{h}_\phi = \mathsf{h}_\phi \circ \phi$ a.e. $[\mu]$. By Proposition 2.3, this yields (i).

Summarizing, we have proved that the conditions (i)–(iv) are equivalent.

(v) $\Rightarrow$ (vi) Since $\mathsf{h}_{\phi^2} = \mathsf{h}_\phi^2$ a.e. $[\mu]$, Lemma 2.2 yields $\mathsf{E}(\mathsf{h}_\phi) = \mathsf{h}_\phi \circ \phi$ a.e. $[\mu]$. Clearly, $\mathsf{h}_{\phi^{n+1}} = \mathsf{h}_{\phi^n} \cdot \mathsf{h}_\phi$ a.e. $[\mu]$ for $n \in \mathbb{Z}_+$ and thus, by Lemma 2.2 again, we have

$$\mathsf{E}(\mathsf{h}_\phi^n) = \mathsf{E}(\mathsf{h}_{\phi^n}) = \mathsf{h}_{\phi^n} \circ \phi = (\mathsf{h}_\phi \circ \phi)^n = \mathsf{E}(\mathsf{h}_\phi)^n \quad \text{a.e. } [\mu] \text{ for every } n \in \mathbb{Z}_+.$$

(vi) $\Rightarrow$ (v) The equality $\mathsf{h}_{\phi^n} = \mathsf{h}_\phi^n$ a.e. $[\mu]$ holds for $n = 0$. Suppose it is valid for a fixed $n \in \mathbb{Z}_+$. Then $\mathsf{E}(\mathsf{h}_{\phi^n}) = \mathsf{h}_\phi^n \circ \phi = \mathsf{h}_{\phi^n} \circ \phi$ a.e. $[\mu]$ which by Lemma 2.2 gives

$$\mathsf{h}_{\phi^{n+1}} = \mathsf{h}_{\phi^n} \cdot \mathsf{h}_\phi = \mathsf{h}_\phi^n \cdot \mathsf{h}_\phi = \mathsf{h}_\phi^{n+1} \quad \text{a.e. } [\mu].$$

(i) $\Rightarrow$ (v) Without loss of generality, we may assume that $\mathsf{h}_\phi(x) < \infty$ for all $x \in X$. Note that $\mathsf{h}_\phi \circ \phi > 0$ a.e. $[\mu]$ (because $\phi^{-1}(\mathsf{N}_\phi) = \{x \in X : \mathsf{h}_\phi(\phi(x)) = 0\}$ and $\mu(\phi^{-1}(\mathsf{N}_\phi)) = \int_X \chi_{\mathsf{N}_\phi} \circ \phi \mathrm{d}\mu = \int_X \chi_{\mathsf{N}_\phi} \mathsf{h}_\phi \mathrm{d}\mu = 0$ with $\mathsf{N}_\phi := \{x \in X : \mathsf{h}_\phi(x) = 0\}$). Hence, by Proposition 2.3, we have $\mathsf{h}_\phi = \mathsf{h}_\phi \circ \phi > 0$ a.e. $[\mu]$. Let $P : X \times \mathfrak{B}(\mathbb{R}_+) \to [0, 1]$ be the $\phi^{-1}(\mathscr{A})$-measurable family of probability measures defined by

$$P(x, \sigma) = \chi_\sigma(\mathsf{h}_\phi(\phi(x))), \quad x \in X, \ \sigma \in \mathfrak{B}(\mathbb{R}_+). \tag{3.3}$$

Then P satisfies (CC) (see e.g., the proof of [5, Proposition 10]). Hence, by Theorem 2.4, we see that for every $n \in \mathbb{Z}_+$,

$$\mathsf{h}_{\phi^n}(x) = \int_{\mathbb{R}_+} t^n P(x, \mathrm{d}t) \stackrel{(3.3)}{=} (\mathsf{h}_\phi \circ \phi)^n(x) = \mathsf{h}_\phi^n(x) \quad \text{for } \mu\text{-a.e. } x \in X.$$

(v) $\Rightarrow$ (iii) Let $\{X_n\}_{n=1}^\infty$ be as in the proof of (iii) $\Rightarrow$ (ii). By (3.2) and the nonsingularity of ϕ, we have

$$\mathsf{h}_\phi \circ \phi \leqslant k \quad \text{a.e. } [\mu] \text{ on } \phi^{-1}(X_k) \text{ for every } k \in \mathbb{N}. \tag{3.4}$$

Now we show that

$$\mathsf{h}_\phi \leqslant k \quad \text{a.e. } [\mu] \text{ on } \phi^{-1}(X_k) \text{ for every } k \in \mathbb{N}. \tag{3.5}$$

For this, note that applying (v) we get

$$\int_{\phi^{-1}(\Delta)} \mathsf{h}_\phi^n \mathrm{d}\mu = \mu((\phi^n)^{-1}(\phi^{-1}(\Delta))) = \int_\Delta \mathsf{h}_\phi^{n+1} \mathrm{d}\mu, \quad \Delta \in \mathscr{A}, \ n \in \mathbb{N}. \tag{3.6}$$

Substituting $\Delta = X_k$ into (3.6) and using (3.2), we obtain

$$\left(\int_{\phi^{-1}(X_k)} \mathsf{h}_\phi^n \mathrm{d}\mu \right)^{1/n} \leqslant (k^{n+1}\mu(X_k))^{1/n}, \quad k, n \in \mathbb{N}. \tag{3.7}$$

By [1, p. 95, Problem 9], this implies (3.5).

It follows from (vi) that $\mathsf{E}(\mathsf{h}_{\phi^n}) = \mathsf{h}_\phi^n \circ \phi$ a.e. $[\mu]$ for all $n \in \mathbb{Z}_+$. Hence, by (2.2),

$$\int_{\phi^{-1}(\Delta)} \mathsf{h}_{\phi^n} \mathrm{d}\mu = \int_{\phi^{-1}(\Delta)} \mathsf{h}_\phi^n \circ \phi \mathrm{d}\mu \tag{3.8}$$

for all $\Delta \in \mathscr{A}$ and $n \in \mathbb{Z}_+$. Fix $k \in \mathbb{N}$ and $\Delta \in \mathscr{A}$ such that $\Delta \subseteq X_k$. In view of (3.4) and (3.5), there exists $E \in \mathscr{A}$ such that $E \subseteq \phi^{-1}(\Delta)$, $\mu(\phi^{-1}(\Delta) \setminus E) = 0$ and

$$\mathsf{h}_\phi(x), \mathsf{h}_\phi(\phi(x)) \in [0, k] \quad \text{for every } x \in E. \tag{3.9}$$

By (3.7), both sides of (3.8) are finite for every $n \in \mathbb{Z}_+$. Therefore, we have

$$\int_E p \circ \mathsf{h}_\phi \mathrm{d}\,\mu = \int_E p \circ \mathsf{h}_\phi \circ \phi \mathrm{d}\,\mu, \quad p \in \mathbb{C}[z], \tag{3.10}$$

where $\mathbb{C}[z]$ is the ring of all complex polynomials in variable z. Let $f \colon [0, k] \to \mathbb{C}$ be a continuous function. By the Weierstrass theorem, there exists a sequence $\{p_n\}_{n=1}^\infty \subseteq \mathbb{C}[z]$ which is convergent uniformly to f on $[0, k]$. Then, by (3.9), both suprema $\sup_E |f \circ \mathsf{h}_\phi \circ \phi - p_n \circ \mathsf{h}_\phi \circ \phi|$ and $\sup_E |f \circ \mathsf{h}_\phi - p_n \circ \mathsf{h}_\phi|$ are less than or equal to $\sup_{[0,k]} |f - p_n|$ for every $n \in \mathbb{N}$. Since, by (3.2), $\mu(E) \leqslant \mu(\phi^{-1}(X_k)) < \infty$, we deduce that $\int_E p_n \circ \mathsf{h}_\phi \circ \phi \mathrm{d}\,\mu$ tends to $\int_E f \circ \mathsf{h}_\phi \circ \phi \mathrm{d}\,\mu$ as $n \to \infty$, and $\int_E p_n \circ \mathsf{h}_\phi \mathrm{d}\,\mu$ tends to $\int_E f \circ \mathsf{h}_\phi \mathrm{d}\,\mu$ as $n \to \infty$. Hence, by (3.10), for every continuous function $f \colon [0, k] \to \mathbb{C}$,

$$\int_E f \circ \mathsf{h}_\phi \mathrm{d}\,\mu = \int_E f \circ \mathsf{h}_\phi \circ \phi \mathrm{d}\,\mu. \tag{3.11}$$

Take an interval $J = [a, b)$ with $a, b \in \mathbb{R}_+$. Then there exists a sequence of continuous functions $f_n \colon [0, k] \to [0, 1]$, $n \in \mathbb{N}$, which converges to $\chi_{J \cap [0,k]}$ pointwise. Therefore, by (3.9), $f_n \circ \mathsf{h}_\phi \circ \phi$ tends to $\chi_{J \cap [0,k]} \circ \mathsf{h}_\phi \circ \phi$ pointwise on E as $n \to \infty$, and $f_n \circ \mathsf{h}_\phi$ tends to $\chi_{J \cap [0,k]} \circ \mathsf{h}_\phi$ pointwise on E as $n \to \infty$. This combined with (3.11) and the Lebesgue dominated convergence theorem shows that the equality

$$\int_{\phi^{-1}(\Delta)} \chi_\sigma \circ \mathsf{h}_\phi \mathrm{d}\,\mu = \int_{\phi^{-1}(\Delta)} \chi_\sigma \circ \mathsf{h}_\phi \circ \phi \mathrm{d}\,\mu \tag{3.12}$$

holds for $\sigma = J \cap [0, k]$. Applying Lemma 2.1 to the Borel measures on $[0, k]$ (with respect to σ) defined by the left-hand and the right-hand sides of (3.12), and to the semi-algebra $\mathscr{P} = \{[a, b) \cap [0, k] \colon a, b \in \mathbb{R}_+\}$, we see that (3.12) holds for every Borel set $\sigma \subseteq [0, k]$.

In view of the above, if $\Delta \in \mathscr{A}$ and $\sigma \in \mathfrak{B}(\mathbb{R}_+)$, then by the Lebesgue monotone convergence theorem and the fact that $X_k \nearrow X$ as $k \to \infty$, we have

$$\int_{\phi^{-1}(\Delta)} \chi_\sigma \circ \mathsf{h}_\phi \mathrm{d}\,\mu = \lim_{k \to \infty} \int_{\phi^{-1}(\Delta \cap X_k)} \chi_{\sigma \cap [0,k]} \circ \mathsf{h}_\phi \mathrm{d}\,\mu$$

$$= \lim_{k \to \infty} \int_{\phi^{-1}(\Delta \cap X_k)} \chi_{\sigma \cap [0,k]} \circ \mathsf{h}_\phi \circ \phi \mathrm{d}\,\mu = \int_{\phi^{-1}(\Delta)} \chi_\sigma \circ \mathsf{h}_\phi \circ \phi \mathrm{d}\,\mu,$$

which together with (2.2) implies (iii). This completes the proof. □

Remark 3.2. Regarding Theorem 3.1, it is worth pointing out that it may happen that the equalities $\mathsf{h}_{\phi^n} = \mathsf{h}_\phi^n$ a.e. $[\mu]$, $n \in \mathbb{Z}_+$, are satisfied though the equality $\mathsf{h}_\phi = \mathsf{h}_\phi \circ \phi$ a.e. $[\mu]$ is not. This shows that the assumption $\overline{\mathcal{D}(C_\phi)} = L^2(\mu)$ in Theorem 3.1 is essential. To demonstrate this, set $X = \mathbb{Z}_+$ and $\mathscr{A} = 2^X$. Let μ be the counting measure on $\mathscr{A}$. Define the nonsingular transformation ϕ of X by $\phi(x) = 0$ for $x \in X$. It is easily seen that $\mathsf{h}_{\phi^n}(x) = 0$ if $x \in X \setminus \{0\}$ and $\mathsf{h}_{\phi^n}(0) = \infty$ for every $n \in \mathbb{N}$. This implies that $\mathsf{h}_{\phi^n}(x) = \mathsf{h}_\phi(x)^n$ for all $x \in X$ and $n \in \mathbb{Z}_+$. However, $\mathsf{h}_\phi \circ \phi(x) = \infty$ for every $x \in X$, and $\mathsf{h}_\phi(x) = 0$ for every $x \in X \setminus \{0\}$.

References

[1] R.B. Ash, Probability and Measure Theory, Harcourt/Academic Press, Burlington, 2000.
[2] M. Sh. Birman, M.Z. Solomjak, Spectral Theory of Selfadjoint Operators in Hilbert Space, D. Reidel Publishing Co., Dordrecht, 1987.
[3] A. Brown, On a class of operators, Proc. Amer. Math. Soc. 4 (1953) 723–728.
[4] P. Budzyński, Z.J. Jabłoński, I.B. Jung, J. Stochel, On unbounded composition operators in L^2-spaces, Ann. Mat. Pura Appl. (2012) http://dx.doi.org/10.1007/s10231-012-0296-4.
[5] P. Budzyński, Z.J. Jabłoński, I.B. Jung, J. Stochel, Unbounded subnormal composition operators in L^2-spaces (submitted for publication). arXiv:1303.6486v1 [math.FA].
[6] C. Burnap, I.B. Jung, A. Lambert, Separating partial normality classes with composition operators, J. Oper. Theory 53 (2005) 381–397.
[7] J.T. Campbell, W.E. Hornor, Seminormal composition operators, J. Oper. Theory 29 (1993) 323–343.
[8] A. Daniluk, J. Stochel, Seminormal composition operators induced by affine transformations, Hokkaido Math. J. 26 (1997) 377–404.
[9] P. Dibrell, J.T. Campbell, Hyponormal powers of composition operators, Proc. Amer. Math. Soc. 102 (1988) 914–918.
[10] M.R. Embry, A. Lambert, Subnormal weighted translation semigroups, J. Funct. Anal. 24 (1977) 268–275.
[11] M. Embry-Wardrop, A. Lambert, Subnormality for the adjoint of a composition operator on L^2, J. Oper. Theory 25 (1991) 309–318.

[12] D. Harrington, R. Whitley, Seminormal composition operators, J. Oper. Theory 11 (1984) 125–135.
[13] Z. Jabłonski, Hyperexpansive composition operators, Math. Proc. Cambridge Philos. Soc. 135 (2003) 513–526.
[14] Z.J. Jabłoński, I.B. Jung, J. Stochel, A non-hyponormal operator generating Stieltjes moment sequences, J. Funct. Anal. 262 (2012) 3946–3980.
[15] A. Lambert, Subnormal composition operators, Proc. Amer. Math. Soc. 103 (1988) 750–754.
[16] A. Lambert, Normal extensions of subnormal composition operators, Michigan Math. J. 35 (1988) 443–450.
[17] W. Mlak, Operators induced by transformations of Gaussian variables, Ann. Polon. Math. 46 (1985) 197–212.
[18] J. Neveu, Bases Mathématiques Du Calcul des Probabilités, Éditeurs, Paris, 1970 (in French).
[19] E. Nordgren, Composition Operators on Hilbert Spaces, in: Lecture Notes in Math., vol. 693, Springer-Verlag, Berlin, 1978, pp. 37–63.
[20] M.M. Rao, Conditional Measures and Applications, in: Pure and Applied Mathematics (Boca Raton), vol. 271, Chapman and Hall/CRC, Boca Raton, 2005.
[21] R.K. Singh, Compact and quasinormal composition operators, Proc. Amer. Math. Soc. 45 (1974) 80–82.
[22] R.K. Singh, J.S. Manhas, Composition Operators on Function Spaces, Elsevier Science Publishers B.V., North-Holland-Amsterdam, 1993.
[23] J. Stochel, Seminormal composition operators on L^2 spaces induced by matrices, Hokkaido Math. J. 19 (1990) 307–324.
[24] J. Stochel, J.B. Stochel, Seminormal composition operators on L^2 spaces induced by matrices: the Laplace density case, J. Math. Anal. Appl. 375 (2011) 1–7.
[25] J. Stochel, F.H. Szafraniec, On normal extensions of unbounded operators, II, Acta Sci. Math. (Szeged) 53 (1989) 153–177.
[26] R. Whitley, Normal and quasinormal composition operators, Proc. Amer. Math. Soc. 70 (1978) 114–118.